If we have shown that for $|\eta| < |\alpha|$, $V_h{}^{\eta}u$ for sufficiently small h have their $L(N_{2-|\alpha|+1_\lambda})$-norms uniformly bounded in h and since the same is true for $V_h{}^{\alpha}\psi$ and $V_h{}^{\zeta}c_{\beta}'$, we may apply relation (3) to $v = V_h{}^{\alpha}u$ and conclude that $\|V_h{}^{\alpha}u\|_{(N-|\alpha|_\lambda)}$ is uniformly bounded in h for all sufficiently small h. But as $h \to 0$, $V_h{}^{\alpha}u$ converges in the distribution sense to $(\partial/\partial x)^{\alpha}u$. If we consider the sequence $V_{2^{-n}h}u$ in $L^2(N_{2-|\alpha|_\lambda})$, then, by the uniform boundedness of the norms, we can find a subsequence converging weakly to an element of $L^2(N_{2-|\alpha|_\lambda})$ which must equal $(\partial/\partial x)^{\alpha}u$ in that neighborhood. Since all the distribution derivatives of u are locally in L^2, our conclusion follows from a well-known theorem of Sobolev.[6]

Remark 2: By a refinement of the argument, we can remove the restriction that $u \in L^2(G)$ and prove regularity for any distribution solution. Similarly, bounds may be obtained for the L^2-norms of u and its derivatives on a compact G_1 in terms of L^2-norms of ψ and its derivatives and any negative Dirichlet norm of u on any G_2 containing $\bar{G}_1$ in its interior.

[1] For a survey of recent results in the elliptic case cf. F. E. Browder, *Communs. Pure and Appl. Math.*, **9**, 351–361, 1956; L. Nirenberg, *Communs. Pure and Appl. Math.*, **9**, 509–529, 1956. For the parabolic case see F. E. Browder, these PROCEEDINGS, **42**, 914, 1956.

[2] L. Hörmander, *Acta math.*, **94**, 161–248, 1955.

[3] The first result of this type was stated by Hörmander.

[4] F. E. Browder, these PROCEEDINGS, **42**, 769–771, 1956.

[5] Hörmander, *op. cit.*, pp. 222–229.

[6] Cf. L. Schwartz, *Théorie des distributions* (Paris, 1951), **2**, 47, Theorem XIX.

TWO RECURSIVELY ENUMERABLE SETS OF INCOMPARABLE DEGREES OF UNSOLVABILITY (SOLUTION OF POST'S PROBLEM, 1944)

BY RICHARD M. FRIEDBERG

HARVARD UNIVERSITY

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Post[1] has questioned the existence of a recursively enumerable set of non-negative integers which is neither recursive nor of the highest degree of unsolvability possible to recursively enumerable sets. This question is now answered by the construction of two recursively enumerable sets of which we prove that neither is recursive in the other and hence that they both satisfy the criteria of Post's question. In the theorem we shall deal not with the sets directly but with their representing functions (functions taking the values 0 and 1, respectively, for members and nonmembers of the sets).[2]

THEOREM I. *There exist two functions f_1 and f_2, both of which represent recursively enumerable sets and neither of which is recursive in the other.*

We shall define f_1 and f_2 by successive approximation through a pair of sequences of functions $f_1{}^0, f_1{}^1, f_1{}^2, \ldots$ and $f_2{}^0, f_2{}^1, f_2{}^2, \ldots$. For each pair of numbers a, e we shall define a number $x_1{}^a(e)$ for which we intend to set $f_1{}^{a'}(x_1{}^a(e)) = 0$ if at some later stage a' $(\geq a)$ of the construction we encounter a y for which

$$T_1{}^{f_2{}^{a'-1}}(e, x_1{}^a(e), y) \ \& \ U(y) = 1, \qquad (1)^3$$

and otherwise to leave $f_1^{a'}(x_1^a(e))$ unchanged as a' increases. Once relation (1) is established, we shall act so as to insure as nearly as possible that it will remain true for all higher a', and hence also with f_2 in place of $f_2^{a'-1}$. Similarly for $x_2^a(e)$, with f_1^a and f_2^a interchanged.

The functions $f_1^a(x)$, $f_2^a(x)$, $x_1^a(e)$, and $x_2^a(e)$ shall be recursive as two-argument functions. We define them as follows.

Case 0: $a = 0$. Let

$$f_1^0(x) = f_2^0(x) = 1, \qquad \text{all } x;$$

$$x_1^0(e) = x_2^0(e) = 2^e, \qquad \text{all } e.$$

Case 1: $a = 2b + 1$. Let e_a be the number of prime divisors of b (so that for any fixed e we have $e_a = e$ for infinitely many a). Two subcases are possible.

Subcase 1.1: $f_1^{a-1}(x_1^{a-1}(e_a)) = 1$ and $(Ey < a)[T_1^{f_2^{a-1}}(e_a, x_1^{a-1}(e_a), y) \;\&\; U(y) = 1]$. Then let

$$f_1^a(x_1^{a-1}(e_a)) = 0;$$

$$x_2^a(e) = 2^e\cdot(2a + 1), \qquad \text{all } e \geqslant e_a;$$

and otherwise let

$$f_1^a = f_1^{a-1}, \qquad f_2^a = f_2^{a-1}, \qquad x_1^a = x_1^{a-1}, \qquad x_2^a = x_2^{a-1}.$$

Subcase 1.2: Otherwise. Then let

$$f_1^a = f_1^{a-1}, \qquad f_2^a = f_2^{a-1}, \qquad x_1^a = x_1^{a-1}, \qquad x_2^a = x_2^{a-1}.$$

Case 2: $a = 2b + 2$. Treat as Case 1 with subscripts 1 and 2 interchanged and with "$e \geqslant e_a$" in line 4 of Subcase 1.1 replaced by "$e > e_a$" in Subcase 1.2.

This completes the definition of the auxiliary functions. Now let $f_1(x) = 0$ or 1 according as $(Ea)(f_1^a(x) = 0)$ or not, and similarly for f_2. Clearly, f_1 and f_2 represent recursively enumerable sets.

Note that the only information about a function f relevant to the statement $T_1^f(e, x, y)$ is information about its values on arguments $u < y$ (because all the relevant arguments occur in a formal expression with Gödel number y). Therefore, the changes made in x_2^a under Subcase 1.1 prevent the falsification of $T_1^{f_2^{a'}}(e_a, x_1^a(e_a), y)$ for any $a' \geq a$ except through an occurrence of Subcase 2.1 with $e_{a'} < e_a$.

The success of the construction depends upon two lemmas.

LEMMA I. *For any given e, $x_1^a(e)$ changes only a finite number of times as a increases through the natural numbers.*

Because there are only finitely many $e < \bar{e}$, the lemma can fail for $e = \bar{e}$ only if, for some fixed $\bar{e}' < \bar{e}$, Subcase 2.1 occurs an infinite number of times with $e_a = \bar{e}'$. Since each such occurrence changes $f_2^a(x_2^a(\bar{e}'))$ from 1 to 0, this in turn requires infinitely many changes of $x_2^a(\bar{e}')$. But this, by similar reasoning, requires infinitely many changes of $x_1^a(e)$ for some fixed $e \leqslant \bar{e}' < \bar{e}$. Therefore, $\bar{e}$ is not the smallest number for which the lemma fails. Thus the lemma is proved by induction.

LEMMA II. *Let $z_1(e)$ be the last (unchanged) value assumed by $x_1^a(e)$ as a increases. Then $(Ey)[T_1^{f_2}(e, z_1(e), y) \;\&\; U(y) = 1] \equiv f_1(z_1(e)) = 0$.*

For if $T_1^{f_2}(e, z_1(e), y)$ & $U(y) = 1$ for some y, then (since, for sufficiently high a, $f_2^a(u) = f_2(u)$, all $u < y$) $f_1^a(z_1(e))$ will eventually be made equal to zero through an occurrence of Subcase 1.1.

Conversely, if $f_1(z_1(e)) = 0$, Subcase 1.1 must have arisen for some a for which $x_1^a(e) = z_1(e)$ and $(Ey)[T_1^{f_2^{a-1}}(e, z_1(e), y)$ & $U(y) = 1]$. No occurrence of Subcase 2.1 with $e_a < e$ can subsequently falsify this latter statement, for such an occurrence would induce a change in $x_1^a(e)$, contrary to the definition of $z_1(e)$. Therefore, the statement remains true with f_2 in place of f_2^{a-1}.

Lemmas I and II hold for any e and as well with f_1 and f_2 interchanged. Therefore, neither f_1 nor f_2 is recursive in the other.[4]

THEOREM II. *Given a set A, there exist two sets not recursive in one another, both enumerable by a procedure recursive in A and both of degree higher than that of A.*

Proof: Change Case 0 in Theorem I to make $f_1^0(2^e) = f_2^0(2^e) = 0$ instead of 1 whenever e belongs to A, and to make $x_1^0(e) = x_2^0(e) = 3 \cdot 2^e$ for all e. Then f_1 and f_2 represent the desired sets.

[1] Emil L. Post, "Recursively Enumerable Sets of Positive Integers and Their Decision Problems," *Bull. Am. Math. Soc.*, **50**, 284–316, 1944.

[2] S. C. Kleene and Emil L. Post ("The Upper Semi-lattice of Degrees of Recursive Unsolvability," *Ann. Math.*, **59**, 379–407, 1954) produce two functions neither of which is recursive in the other. The present paper adapts their method to the restriction that both functions represent recursively enumerable sets. The gist of this adaptation was presented by title at the February 25, 1956, meeting of the American Mathematical Society and has been abstracted (R. Friedberg, *Bull. Am. Math. Soc.*, **62**, 260, 1956, Abstr. 362).

[3] U is a recursive function, and T_1^f is a predicate which is recursive if f is, such that $T_1^f(e, x, y)$ means that e is the Gödel number of a formal procedure for calculating one function, given another; that y is the Gödel number of a formal application of this procedure with f as the given function; and that this application yields the value $U(y)$ for the calculated function on the argument x. (See S. C. Kleene, *Introduction to Metamathematics* [New York: D. Van Nostrand Co., 1952], pp. 276–278 and 288–291.) Hence f_1 cannot be recursive in f_2 if for each e there is an x such that

$$f_1(x) \neq 1 \equiv (Ey)[T_1^{f_2}(e, x, y) \; \& \; U(y) = 1].$$

[4] The nonconstructiveness of Lemma I, which demonstrates that z_1 is defined for all arguments without telling us how to calculate it, is a necessary feature of the construction. For if z_1 were recursive, f_1 would represent a creative set and would therefore, by a theorem of J. R. Myhill ("Creative Sets," *Z. math. Logik u. Grundlagen Math.*, **1**, 97–108, 1955), be of the highest degree possible to recursively enumerable sets.

A full exposition of this theorem is to appear in two forthcoming texts: H. Rogers, *Theory of Recursive Functions and Effective Computability* (mimeo), MIT Math. Dept., Cambridge, 1957; and J. C. E. Dekker and J. R. Myhill, *Recursion Theory*.